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PARAMETER ESTIMATION THEORY AND SOME APPLICATIONS OF THE THEORY TO RADAR MEASUREMENTS

by

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Abstract: The general theory of parameter estimation is developed using the inverse probability approach. Where the measurements are perturbed by additive Gaussian noise and when the received information is sufficient to determine the parameters of interest rather accurately, it is shown that an optimum method of processing redundant data based on the maximum likelihood approach reduces approximately to the solution of k nonlinear equations in the k unknown parameters. An expression is derived for the resulting error moment matrix of the parameters. It is shown that this same moment matrix for a minimum variance estimate is obtained by using results derived by Cramér. The theory is illustrated by applying it to several radar measurement problems of interest.

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PARAMETER ESTIMATION THEORY AND SOME APPLICATIONS OF THE THEORY TO RADAR MEASUREMENTS

THE METHOD OF INVERSE PROBABILITY*

Suppose we have some system about which we are to receive information and that the state of this system is characterized by a set of k parameters. We receive a message or make measurements on the system, which are generally perturbed by noise, and on the basis of this information and our own *a priori* information, we must arrive at some optimum estimate about the state of the system.

We make the following definitions:

$$\gamma = [\gamma_1, \gamma_2, \cdots, \gamma_k] = \text{true parameters}$$

$$\hat{\gamma} = [\hat{\gamma}_1, \hat{\gamma}_2, \cdots, \hat{\gamma}_k] = \text{estimated parameter values}$$

$$X = X(\gamma) = [x_1, x_2, \cdots, x_n] = \text{noiseless message}$$

$$Y = [y_1, y_2, \cdots, y_n] = \text{message perturbed by noise}$$

The state of the system is thus characterized by a point or vector in a k-dimensional parameter space. The quantities γ, $\hat{\gamma}$, X, and Y will denote column vectors, though vector signs have been omitted for simplicity of notation. A bar will be used to denote average or expected value, and τ will denote transpose.

In the situation where the number of measurements n is less than the number of parameters k, a noiseless message X is not sufficient to determine γ uniquely. When $n = k$, the message X is in general just sufficient to determine γ and we have the so-called *minimum data* case. When $n > k$, γ is in general over-specified by X and we have the so-called *redundant data* case.

The received message or measurement Y differs from X because of noise, and the effect of this noise is characterized by the conditional probability density $P(Y/X)$ [or $P(Y/\gamma)$ since $X = X(\gamma)$]. Given the message Y we can express the *a posteriori* probability density over γ in terms of $P(Y/\gamma)$ and the *a priori* probability density $P(\gamma)$ with the aid of Bayes' theorem.

$$P(\gamma/Y) = \frac{P(Y/\gamma)P(\gamma)}{P(Y)}$$

$$\text{where} \quad P(Y) = \int P(Y/\gamma)P(\gamma)\,d\gamma \tag{1}$$

$$\text{and} \int [\quad]\,d\gamma \text{ stands for} \int \overset{k}{\cdots} \int [\quad]\,d\gamma_1\, d\gamma_2 \cdots d\gamma_k$$

* Most of the theory of this paper follows the material presented in Reference 1.

Since we are interested in the dependence of $P(\gamma/Y)$ on γ, we shall find it convenient to ignore factors which are independent of γ and write

$$P(\gamma/Y) \sim P(Y/\gamma)P(\gamma) \tag{2}$$

where ~ stands for "is proportional to"

It is convenient to carry through calculations on probability densities in terms of proportionalities, recognizing that the constant of proportionality can always be inserted later simply by requiring that the integrated probability be equal to unity.

The above formulas contain, in principle, all one needs to know about the parameter estimation problem. The value of γ which maximizes $P(Y/\gamma)$ is called the *maximum likelihood* estimate. When $P(\gamma)$ can be considered to be constant over the region of interest, the maximum likelihood estimate also maximizes $P(\gamma/Y)$. We proceed now to specialize the theory to the case of additive Gaussian noise.

SPECIALIZATION OF THE THEORY TO THE CASE OF ADDITIVE GAUSSIAN NOISE

Assume that the error Y-X results from additive Gaussian noise characterized by an $n \times n$ moment matrix M. Then, $P(Y/\gamma)$ is a multidimensional normal distribution of the form*

$$\begin{gathered} P(Y/\gamma) = P[Y/X(\gamma)] = P[Y - X(\gamma)] \sim e^{-\frac{1}{2}[Y - X(\gamma)]^T M^{-1} [Y - X(\gamma)]} \\ \text{where } M = \overline{[Y - X(\gamma)][Y - X(\gamma)]^T} \end{gathered} \tag{3}$$

Substituting this expression for $P(Y/\gamma)$ into Eq. 2, we obtain

$$P(\gamma/Y) \sim P(Y/\gamma)P(\gamma) \sim P(\gamma)e^{-\frac{1}{2}[Y - X(\gamma)]^T M^{-1} [Y - X(\gamma)]} \tag{4}$$

If the measurements are capable of leading to a rather accurate determination of γ, the argument of the exponential will be sharply peaked around the correct value. If $P(\gamma)$ is a very slowly varying function of γ over the region of interest, we may justifiably treat it as a constant of proportionality. With this assumption, the estimate $\hat{\gamma}$ which maximizes $P(\gamma/Y)$ also maximizes $P(Y/\gamma)$ and is therefore a maximum likelihood estimate. In order for $\hat{\gamma}$ to be a maximum likelihood estimate, it must satisfy the vector equation

$$\frac{\partial}{\partial \gamma}\left\{\tfrac{1}{2}[Y - X(\gamma)]^T M^{-1}[Y - X(\gamma)]\right\} = -\left[\frac{\partial X(\gamma)}{\partial \gamma}\right]^T M^{-1}[Y - X(\gamma)] = 0 \tag{5}$$

where it is understood that $\frac{\partial X(\gamma)}{\partial \gamma}$ is an $n \times k$ matrix with ij^{th} element $\frac{\partial X_i(\gamma)}{\partial \gamma_j}$. The solution for $\hat{\gamma}$ is seen to involve the solution of k generally nonlinear equations in the k unknown parameters.

* For a discussion of the multidimensional normal distribution see, for example, Reference 2.

Situations can arise in which more than one value of γ may satisfy these equations. Some of the values represent local maxima or saddle points, or perhaps even local minima of $P(\gamma/Y)$. In these situations we must find other means for deciding which one of the values of γ corresponds to an absolute maximum of $P(\gamma/Y)$.

Using the earlier assumption that Y leads to a rather accurate determination of γ, and assuming that the functional dependence of X on γ is sufficiently well behaved, we can approximate in the region of small errors

$$X(\gamma) \approx X(\hat{\gamma}) + \left.\frac{\partial X(\gamma)}{\partial \gamma}\right|_{\hat{\gamma}} d\gamma \qquad \text{where } d\gamma = \gamma - \hat{\gamma} \tag{6}$$

where higher order terms have been neglected. Using this expression in Eq. 4 for $P(\gamma/Y)$, we have

$$\begin{aligned} P(\gamma/Y) &\sim e^{-\frac{1}{2}\left[Y - X(\hat{\gamma}) - \left.\frac{\partial X(\gamma)}{\partial \gamma}\right|_{\hat{\gamma}} d\gamma\right]^{T} M^{-1} \left[Y - X(\hat{\gamma}) - \left.\frac{\partial X(\gamma)}{\partial \gamma}\right|_{\hat{\gamma}} d\gamma\right]} \\ &\sim e^{\left[\left\{-\frac{1}{2}[Y - X(\hat{\gamma})]^{T} M^{-1} [Y - X(\hat{\gamma})]\right\} + \left\{d\gamma^{T}\left[\frac{\partial X(\gamma)}{\partial \gamma}\right]^{T}_{\hat{\gamma}} M^{-1}[Y - X(\hat{\gamma})]\right\} - \left\{\frac{1}{2} d\gamma^{T}\left[\frac{\partial X(\gamma)}{\partial \gamma}\right]^{T}_{\hat{\gamma}} M^{-1} \left.\frac{\partial X(\gamma)}{\partial \gamma}\right|_{\hat{\gamma}} d\gamma\right\}\right]} \end{aligned} \tag{7}$$

The first term in the exponent of e does not depend on γ and therefore represents just a constant of proportionality. The second term in the exponent is zero by virtue of Eq. 5. Eq. 7 then reduces to

$$\begin{aligned} P(\gamma/Y) &\sim e^{-\frac{1}{2} d\gamma^{T} \Gamma^{-1} d\gamma} \\ \text{where } \Gamma^{-1} &= \left[\frac{\partial X(\gamma)}{\partial \gamma}\right]^{T}_{\hat{\gamma}} M^{-1} \left.\frac{\partial X(\gamma)}{\partial \gamma}\right|_{\hat{\gamma}} \\ \text{or } (\Gamma^{-1})_{ij} &= \left[\frac{\partial X(\gamma)}{\partial \gamma_i}\right]^{T}_{\hat{\gamma}} M^{-1} \left.\frac{\partial X(\gamma)}{\partial \gamma_j}\right|_{\hat{\gamma}} \end{aligned} \tag{8}$$

Our assumptions have led us to a $P(\gamma/Y)$ which is a multidimensional normal distribution with a $k \times k$ moment matrix Γ. The variance in the i^{th} parameter is Γ_{ii}^{2}. In order that $P(\hat{\gamma}/Y)$ be a relative maximum, it is necessary that the quadratic form $d\gamma^{T} \Gamma^{-1} d\gamma$ be always positive for $d\gamma \neq 0$. It follows that the matrix Γ^{-1} (and also Γ) must be positive definite.*

It is interesting to compare the result obtained here for the error moment matrix of the parameters with that obtained by Cramér. Cramér (Reference 2) has shown that a *joint efficient* minimum variance estimate has an error moment matrix Γ which satisfies

* For a discussion of positive definiteness, see, for example, p. 273 of Reference 3.

$$(\Gamma^{-1})_{ij} = \overline{\frac{\partial \log P(Y/\gamma)}{\partial \gamma_i} \; \frac{\partial \log P(Y/\gamma)}{\partial \gamma_j}} \tag{9}$$

where, as before, the bar denotes an average. Inserting $P(Y/\gamma)$ from Eq. 3 into this expression, we have

$$\frac{\partial \log P(Y/\gamma)}{\partial \gamma_i} = \left[\frac{\partial X(\gamma)}{\partial \gamma_i}\right]^T M^{-1} [Y - X(\gamma)]$$

$$\frac{\partial \log P(Y/\gamma)}{\partial \gamma_j} = \left[\frac{\partial X(\gamma)}{\partial \gamma_j}\right]^T M^{-1} [Y - X(\gamma)] = [Y - X(\gamma)]^T M^{-1} \frac{\partial X(\gamma)}{\partial \gamma_j}. \tag{10}$$

Then

$$\begin{aligned}(\Gamma^{-1})_{ij} &= \overline{\left[\frac{\partial X(\gamma)}{\partial \gamma_i}\right]^T M^{-1} [Y - X(\gamma)][Y - X(\gamma)]^T M^{-1} \frac{\partial X(\gamma)}{\partial \gamma_j}} \\ &= \left[\frac{\partial X(\gamma)}{\partial \gamma_i}\right]^T M^{-1} \overline{[Y - X(\gamma)][Y - X(\gamma)]^T} M^{-1} \frac{\partial X(\gamma)}{\partial \gamma_j} \\ &= \left[\frac{\partial X(\gamma)}{\partial \gamma_i}\right]^T M^{-1} \frac{\partial X(\gamma)}{\partial \gamma_j}\end{aligned} \tag{11}$$

or

$$\Gamma^{-1} = \left[\frac{\partial X(\gamma)}{\partial \gamma}\right]^T M^{-1} \frac{\partial X(\gamma)}{\partial \gamma}$$

4

If we take for the value of γ in this equation the best estimate $\hat{\gamma}$, we see that this expression for Γ agrees with that obtained earlier (Eq. 8).

EXTENSION OF THE THEORY TO INCLUDE A *PRIORI* INFORMATION ABOUT THE PARAMETERS

Suppose that our *a priori* knowledge about the parameters may be approximately characterized by a multidimensional normal probability distribution.

$$P(\gamma) \sim e^{-\frac{1}{2}(\gamma-\gamma^*)^T \mathcal{P}^{-1} (\gamma-\gamma^*)} \tag{12}$$

where $\mathcal{P}$ is the moment matrix of the distribution and $\gamma^* = [\gamma_1^*, \gamma_2^*, \cdots, \gamma_k^*]$ is the most probable *a priori* value of γ. Then

$$P(\gamma/Y) \sim P(\gamma)P(Y/\gamma) \sim e^{-\frac{1}{2}(\gamma-\gamma^*)^T \mathcal{P}^{-1} (\gamma-\gamma^*) - \frac{1}{2}[Y - X(\gamma)]^T M^{-1} [Y - X(\gamma)]} \tag{13}$$

If $\hat{\gamma}$ is that value of γ which maximizes $P(\gamma/Y)$, $\hat{\gamma}$ must satisfy

$$\frac{\partial}{\partial \gamma}\left[\tfrac{1}{2}(\gamma - \gamma^*)^T \mathcal{P}^{-1} (\gamma - \gamma^*) + \tfrac{1}{2}[Y - X(\gamma)]^T M^{-1} [Y - X(\gamma)]\right] = 0 \tag{14}$$

This leads to a generalization of Eq. 5 which $\hat{\gamma}$ must satisfy.

$$\mathcal{P}^{-1}(\gamma-\gamma^*) - \left[\frac{\partial X(\gamma)}{\partial \gamma}\right]^T M^{-1}[Y - X(\gamma)] = 0 \qquad \text{when } \gamma = \hat{\gamma} \tag{15}$$

Once again the optimum processing of received data to determine $\hat{\gamma}$ involves the solution of k nonlinear equations in the k unknown parameters. As before, we approximate

$$X(\gamma) \approx X(\hat{\gamma}) + \left.\frac{\partial X(\gamma)}{\partial \gamma}\right|_{\hat{\gamma}} d\gamma, \qquad d\gamma = \gamma - \hat{\gamma} \tag{16}$$

Substituting this approximation into Eq. 13, using Eq. 15, and ignoring constants of proportionality, we have a result analogous to Eq. 8

$$P(\gamma/Y) \sim e^{-\frac{1}{2} d\gamma^T \Gamma^{-1} d\gamma}$$

$$\text{where } \Gamma^{-1} = \mathcal{P}^{-1} + \left[\frac{\partial X(\gamma)}{\partial \gamma}\right]^T_{\hat{\gamma}} M^{-1} \left.\frac{\partial X(\gamma)}{\partial \gamma}\right|_{\hat{\gamma}} \tag{17}$$

$$\text{or } (\Gamma^{-1})_{ij} = (\mathcal{P}^{-1})_{ij} + \left[\frac{\partial X(\gamma)}{\partial \gamma_i}\right]^T_{\hat{\gamma}} M^{-1} \left.\frac{\partial X(\gamma)}{\partial \gamma_j}\right|_{\hat{\gamma}}$$

SPECIALIZATION OF THE THEORY TO DIAGONAL M

Let us consider briefly the situation which occurs very frequently in which the error in each measurement is statistically independent of the errors in the other measurements. Denoting the variance of the error in the u^{th} measurement by σ_u^2, M has the following diagonal form 5

$$M = \begin{bmatrix} \sigma_1^2 & & & \bigcirc \\ & \sigma_2^2 & & \\ & & \ddots & \\ \bigcirc & & & \sigma_n^2 \end{bmatrix} \quad \text{and} \quad M^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & & & \bigcirc \\ & \frac{1}{\sigma_2^2} & & \\ & & \ddots & \\ \bigcirc & & & \frac{1}{\sigma_n^2} \end{bmatrix} \tag{18}$$

Equation 5 for $\hat{\gamma}$ then reduces to

$$\sum_{u=1}^{n} \frac{\partial X_u}{\partial \gamma_i} \frac{(Y_u - X_u)}{\sigma_u^2} = 0 \qquad (i = 1, 2, \cdots, k) \tag{19}$$

Equation 8 for Γ reduces to

$$(\Gamma^{-1})_{ij} = \sum_{u=1}^{n} \frac{\partial X_u}{\partial \gamma_i} \frac{1}{\sigma_u^2} \frac{\partial X_u}{\partial \gamma_j} \tag{20}$$

SPECIFIC APPLICATIONS OF THE THEORY

Narrowband Signal With Unknown Amplitude and Carrier Phase in the Presence of Additive Gaussian Noise

Following the appropriate narrowband filtering operation on a narrowband signal imbedded in wideband Gaussian noise, the output at a particular moment can be resolved into quadrature components Y_1 and Y_2 given by

$$\begin{aligned} Y_1 &= A\cos\phi + n_1 \\ Y_2 &= A\sin\phi + n_2 \end{aligned} \tag{21}$$

where A represents the unknown amplitude of the output, ϕ the unknown carrier phase, and n_1 and n_2 the independent noise outputs of the quadrature channels each with variance σ^2 (Reference 4). The noise components n_1 and n_2 have a two-dimensional normal distribution with diagonal moment matrix M.

$$M = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} \tag{22}$$

We identify $X_1(\gamma) = \gamma_1 \cos \gamma_2$, $X_2(\gamma) = \gamma_1 \sin \gamma_2$, where $A = \gamma_1$ and $\phi = \gamma_2$. Γ^{-1} can be obtained from Eq. 20,

$$\Gamma^{-1} = \frac{1}{\sigma^2}\begin{bmatrix} \left(\frac{\partial X_1}{\partial \gamma_1}\right)^2 + \left(\frac{\partial X_2}{\partial \gamma_1}\right)^2 & \frac{\partial X_1}{\partial \gamma_1}\frac{\partial X_1}{\partial \gamma_2} + \frac{\partial X_2}{\partial \gamma_1}\frac{\partial X_2}{\partial \gamma_2} \\ \frac{\partial X_1}{\partial \gamma_1}\frac{\partial X_1}{\partial \gamma_2} + \frac{\partial X_2}{\partial \gamma_1}\frac{\partial X_2}{\partial \gamma_2} & \left(\frac{\partial X_1}{\partial \gamma_2}\right)^2 + \left(\frac{\partial X_2}{\partial \gamma_2}\right)^2 \end{bmatrix} = \frac{1}{\sigma^2}\begin{bmatrix} 1 & 0 \\ 0 & A^2 \end{bmatrix} \tag{23}$$

from which it follows

$$\Gamma = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2/A^2 \end{bmatrix} \tag{24}$$

We then have for δ_A and δ_ϕ, the rms errors in estimating A and ϕ, respectively

$$\begin{aligned} \delta_A &= \Gamma_{11}^{1/2} = \sigma \\ \delta_\phi &= \Gamma_{22}^{1/2} = \sigma/A \text{ (radians)} \end{aligned} \tag{25}$$

Because Γ is diagonal, the errors in A and ϕ are uncoupled. Note that the assumption of small errors which led to Eq. 6 restricts the validity of the expression for δ_ϕ to the case where $\sigma << A$.

The quantity A^2/σ^2 is the (peak) signal-to-noise ratio after the narrowband filter. For matched filter detection of a pulsed signal immersed in white Gaussian noise, A^2/σ^2 is given by

$$\frac{A^2}{\sigma^2} = \frac{2E}{N_o} \tag{26}$$

where E is the signal energy and N_o is the noise power per unit bandwidth at the input to the matched filter (see, for example, Reference 5). δ_ϕ, expressed in terms of $2E/N_o$, is

$$\delta_\phi = \frac{1}{\sqrt{2E/N_o}} \tag{27}$$

For a radar pulse with wavelength λ, the rms error δ_ϕ translates into an rms error in measuring range, δ_r, given by

$$\delta_r = \frac{\lambda \delta_\phi}{4\pi} = \frac{\lambda}{4\pi\sqrt{2E/N_o}} \tag{28}$$

This range error is usually extremely small because it corresponds to a range error of only a fraction of a wavelength. Taken by itself, this measurement of range based on a measurement of carrier phase is highly ambiguous because ϕ is measured modulo 2π. Situations exist, however, where it is possible to combine several ambiguous but accurate measurements of range in order to obtain a nonambiguous determination of velocity or acceleration.

It is interesting to compare the above expression for range accuracy with a range accuracy calculation based on Woodward's formula (Reference 6). With approximations similar to the ones which have been made in this paper, Woodward has derived the following general formula for rms range accuracy:

$$\delta_r = \frac{c}{2}\delta_t = \frac{c}{2\beta\sqrt{2E/N_o}} \tag{29}$$

where δ_r = rms range error

δ_t = rms time error

c = velocity of light

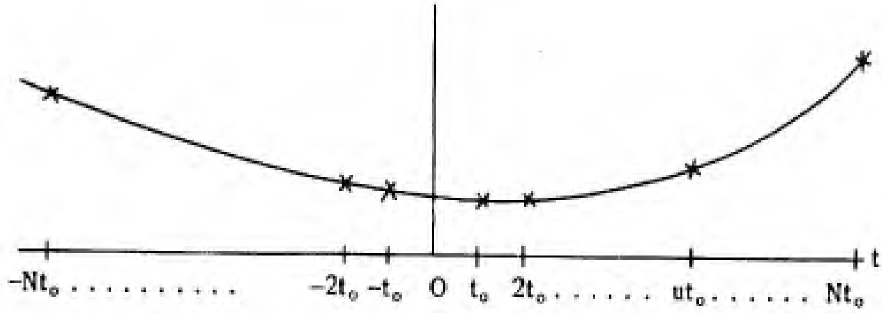
β = 2π (rms bandwidth of signal)

This formula is usually used to calculate "coarse" range accuracy available from a radar pulse by inserting into the formula a β which is calculated by taking the mean or carrier frequency as reference. As Woodward shows, when the bandwidth is calculated with zero frequency as reference, we obtain a range accuracy corresponding to the "fine structure" information in the waveform. For a very narrowband signal with carrier frequency f, $\beta \approx 2\pi f$. When this value of β is inserted into Woodward's formula (Eq. 29), we obtain exactly the same expression for range accuracy as that obtained earlier (Eq. 28), as expected.

Best Fit of a 2nd-Degree Polynomial to a Set of Independent, Equally Spaced Measurements With Equal Variance

We shall take the number of measurements n to be odd so that we can set $n = 2N + 1$. (The case of an even number of measurements is not materially different.) The time origin is taken at the center of the measurement interval. Each of the measurement errors is independent and with variance σ^2.

$$x(t) = \gamma_1 + \gamma_2 t + \gamma_3 t^2$$



$2N + 1$ = total number of measurements

t_o = time interval between measurements

$X_u = \gamma_1 + \gamma_2 ut_o + \gamma_3 u^2 t_o^2$ = noiseless measurement at time ut_o

Y_u = measurement at time ut_o with noise

σ^2 = variance of the error in each measurement

$\gamma_1, \gamma_2, \gamma_3$ = three unknown signal parameters

From Eq. 19, the set of equations to be solved for the parameters is

$$\sum_{u=-N}^{N} \frac{\partial X_u}{\partial \gamma_i} (Y_u - X_u) = 0 \qquad (i = 1, 2, \cdots, k) \tag{30}$$

Using

$$\frac{\partial X_u}{\partial \gamma_1} = 1, \ \frac{\partial X_u}{\partial \gamma_2} = ut_o, \ \frac{\partial X_u}{\partial \gamma_3} = u^2 t_o^2 \tag{31}$$

in Eq. 30, we obtain after some algebra

$$\begin{aligned}
\gamma_1 \left(\sum_u (1)\right) + \gamma_2 (0) \quad &+ \gamma_3 \left(t_o^2 \sum_u u^2\right) = \sum_u Y_u \\
\gamma_1 (0) \quad + \gamma_2 \left(t_o \sum_u u^2\right) &+ \gamma_3 (0) \quad = \sum_u uY_u \\
\gamma_1 \left(\sum_u u^2\right) + \gamma_2 (0) \quad &+ \gamma_3 \left(t_o^2 \sum_u u^4\right) = \sum_u u^2 Y_u
\end{aligned} \tag{32}$$

These equations are fortunately linear and one can easily solve for the parameters in terms of the three quantities $\sum_u Y_u$, $\sum_u uY_u$, $\sum_u u^2Y_u$. We use the relations

$$\sum_{u=-N}^{N} (1) = 2N + 1 = n$$

$$\sum_{u=-N}^{N} u^2 = \frac{N}{3}(N+1)(2N+1) \rightarrow \frac{n^3}{12} \ (n>>1) \tag{33}$$

$$\sum_{u=-N}^{N} u^4 = \frac{N}{15}(N+1)(2N+1)(3N^2+3N-1) \rightarrow \frac{n^5}{80} \ (n>>1)$$

Solving for γ_1, γ_2, γ_3, we obtain

$$\gamma_1 = \frac{9}{4n}\sum_u Y_u - \frac{15}{n^3}\sum_u u^2Y_u$$

$$\gamma_2 = \frac{12}{n^3 t_o}\sum_u uY_u \tag{34}$$

$$\gamma_3 = -\frac{15}{n^3 t_o^2}\sum_u Y_u + \frac{180}{n^5 t_o^2}\sum_u u^2Y_u$$

For this case Eq. 20 for $(\Gamma^{-1})_{ij}$ can be written

$$(\Gamma^{-1})_{ij} = \frac{1}{\sigma^2}\sum_u \frac{\partial X_u}{\partial \gamma_i}\frac{\partial X_u}{\partial \gamma_j} \tag{35}$$

Using Eq. 31 in Eq. 35 and assuming n is large, as before,

$$\Gamma^{-1} = \frac{n}{\sigma^2}\begin{bmatrix} 1 & 0 & \frac{n^2t_o^2}{12} \\ 0 & \frac{n^2t_o^2}{12} & 0 \\ \frac{n^2t_o^2}{12} & 0 & \frac{n^4t_o^4}{80} \end{bmatrix} \tag{36}$$

Then

$$\Gamma = \frac{\sigma^2}{n}\begin{bmatrix} 9/4 & 0 & \frac{15}{n^2t_o^2} \\ 0 & \frac{12}{n^2t_o^2} & 0 \\ \frac{15}{n^2t_o^2} & 0 & \frac{180}{n^4t_o^4} \end{bmatrix} \tag{37}$$

This moment matrix has its zeros placed so that there is no coupling between errors in the coefficients of the even and odd terms of the polynomial $x(t) = \gamma_1 + \gamma_2 t + \gamma_3 t^2$. It follows that the rms error in estimating γ_2 does not depend on whether γ_1 or γ_3 is known, but the error in γ_3 does depend on whether γ_1 is known, and conversely. A similar argument applies also for polynomials of higher degree. The rms error in γ_1, γ_2, γ_3, denoted by δ_{γ_1}, δ_{γ_2}, and δ_{γ_3}, are given by

$$
\begin{aligned}
\delta_{\gamma_1} &= \Gamma_{11}^{1/2} = \frac{3\sigma}{2n^{1/2}} \\
\delta_{\gamma_2} &= \Gamma_{22}^{1/2} = \frac{2\sqrt{3}\,\sigma}{n^{3/2}\,t_o} \\
\delta_{\gamma_3} &= \Gamma_{33}^{1/2} = \frac{6\sqrt{5}\,\sigma}{n^{5/2}\,t_o^2}
\end{aligned}
\tag{38}
$$

We shall use these formulas later to derive expressions for the rms error in estimating radial velocity, radial acceleration, and angular rate from radar measurements.

It is interesting to compare $\delta_{\gamma_1}^2$ with the value which it would have had if γ_3 were a known constant, $\delta_{\gamma_1}^2$ (γ_3 known). This variance is easily calculated by ignoring all rows and columns in Γ^{-1} except the first row and column. This calculation gives simply

$$
\delta_{\gamma_1}^2\,(\gamma_3 \text{ known}) = \frac{1}{(\Gamma^{-1})_{11}} = \frac{\sigma^2}{n}
\tag{39}
$$

and

$$
\frac{\delta_{\gamma_1}^2}{\delta_{\gamma_1}^2\,(\gamma_3 \text{ known})} = \frac{9}{4}
\tag{40}
$$

In a similar manner

$$
\frac{\delta_{\gamma_3}^2}{\delta_{\gamma_3}^2\,(\gamma_1 \text{ known})} = \frac{9}{4}
\tag{41}
$$

As one would expect, the introduction of unknown parameters degrades the measurement accuracy in both cases.

Accuracy of Measuring Radial Velocity and Radial Acceleration From Fine Range Measurements

We assume that during a short interval of time the range as a function of time can be written

$$
r(t) = r_o + vt + \tfrac{1}{2}at^2
\tag{42}
$$

where the constants v and a are to be determined. r_o is assumed to be unknown also. The radar obtains on each pulse a fine range accuracy given by Eq. 28

$$\delta_r = \frac{\lambda}{4\pi\sqrt{2E/N_o}} \tag{43}$$

where E is the returned signal energy on *one* pulse. It is assumed that a series of reflected radar pulses results in a series of $n(n>>1)$ independent range measurements over a time interval T, each with rms error $\sigma = \delta_r$, and with uniform spacing t_o. We have $t_o = T/(n-1) \approx T/n$. Combining Eqs. 38 and 43, we have*

$$\delta_v = \delta_{\gamma_2} = \frac{2\sqrt{3}\,\delta_r}{n^{3/2} t_o} = \frac{\sqrt{3}\,\lambda}{2\pi T\sqrt{2nE/N_o}} = \frac{.276\lambda}{T\sqrt{2nE/N_o}} \tag{44}$$

and

$$\delta_a = 2\delta_{\gamma_3} = \frac{12\sqrt{5}\,\delta_r}{n^{5/2} t_o^2} = \frac{3\sqrt{5}\,\lambda}{\pi T^2\sqrt{2nE/N_o}} = \frac{2.14\lambda}{T^2\sqrt{2nE/N_o}} \tag{45}$$

It is important to keep in mind that these formulas do not account for possible ambiguities in the parameter estimates.

Accuracy of Measuring Angular Rate

We assume that the angle-of-arrival, Θ, may be approximated by

$$\Theta = \Theta_o + \omega t \tag{46}$$

over a short measurement interval of interest, where Θ_o and ω are unknown constants which are to be determined. The general formula for the maximum angular accuracy in estimating angle of arrival is**

$$\delta\Theta = \frac{\lambda}{\mathfrak{L}\sqrt{2E/N_o}} \tag{47}$$

where $\mathfrak{L} = 2\pi \times$ rms length of the antenna (taken about the centroid) and E and N_o are defined as before. We assume, as before, that we have a series of $n(n>>1)$ independent, equally spaced measurements of angle over a time interval $T \approx nt_o$. Combining Eqs. 38 and 47, we have

$$\delta_\omega = \frac{2\sqrt{3}\,\delta\Theta}{n^{3/2} t_o} = \frac{2\sqrt{3}\,\lambda}{\mathfrak{L}T\sqrt{2nE/N_o}} = \frac{3.46\lambda}{\mathfrak{L}T\sqrt{2nE/N_o}} \tag{48}$$

For the case in which the antenna aperture consists of two disjoint coplanar apertures with equal areas and separated by a distance D, which is large compared to the linear dimensions

* The expression for δ_v agrees with that derived in Reference 7.

** The justification of this formula is indicated in Reference 8.

of each aperture, we have $\mathcal{L} \approx \pi D$ and

$$\delta_\omega = \frac{2\sqrt{3}\,\lambda}{\pi\, DT\sqrt{2nE/N_o}} = \frac{1.10\lambda}{DT\sqrt{2nE/N_o}} \tag{49}$$

As before, we must emphasize that this formula does not take into account the possibility of ambiguities in the estimation of angular rate.

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